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Mixing of two-level unstable systems

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Abstract. Unstable particles can be consistently described in the framework of quantum field theory. Starting from the full S matrix amplitudes of $B^+ \to (2\pi,\ 3\pi)l^+\nu$ decays as examples of processes where the $\rho-\omega$ resonances dominate, we propose a prescription for the mixing of two "quasi-physical" unstable states that differs from the one obtained from the diagonalization of the $M - \mathrm{i}\Gamma/2$ non-Hermitian Hamiltonian. We discuss some important consequences for CP violation in the $K_L - K_S$ system.

The $\rho - \omega$ and $K_L - K_S$ mesons are two prime examples of two-level mixed systems useful for studying the important properties of quantum mechanics and the fundamental interactions of unstable particles. The effects of isospin breaking, in the case of the $\rho - \omega$ system, and CP violation, in the case of the $K_L - K_S$ system, convert the corresponding eigenstates (ρ^I, ω^I) or (K_1, K_2) into physical eigenstates (ρ, ω) and (K_L, K_S). These systems allow us to study the violation of fundamental symmetries where the effects of instabilities play an essential role.

Unstable particles can be consistently treated only in the framework of quantum field theory [1,2]. They cannot be described by asymptotic states in the calculation of physical S-matrix amplitudes. Instead, they are associated to propagation amplitudes (propagators) between their production and decay locations, and one cannot detach them from these mechanisms to extract truncated amplitudes. In quantum field theory, unstable particles or resonances are special cases of nonperturbative phenomena obtained from a full re-summation of perturbative bubble graphs [1,2]. In addition, the space-time behaviour of the amplitudes for production and decay of resonances obey, in extremely good approximation, the celebrated exponential decay law and the covariance properties for the time-evolution amplitudes [2].

The conventional quantum-mechanical treatment of symmetry breaking in two-level unstable systems consists in finding the eigenstates that diagonalize a non-Hermitian effective Hamiltonian of the form $H = M - \mathrm{i}\Gamma/2$ [3,4], where M and Γ are 2×2 Hermitian matrices describing the mass and decay properties [5] of the unstable states. H governs the time evolution of the so-called physical eigenstates, which at initial time are given by

$$|X\rangle = |X^s\rangle + \epsilon|Y^s\rangle \ , \qquad (1)$$

$$|Y\rangle = |Y^s\rangle - \epsilon|X^s\rangle \ , \qquad (2)$$

$$\epsilon = \frac{\langle X^s|H^{\mathrm{SB}}|Y^s\rangle}{m_X - m_Y + \frac{\mathrm{i}}{2}(\Gamma_Y - \Gamma_X)} \ . \qquad (3)$$

Here $|Z^s\rangle$ denotes an interaction eigenstate, m_Z (Γ_Z) is the mass (decay width) of the unstable state, and ϵ is the mixing parameter due to symmetry breaking. H^{SB} is the symmetry-breaking Hamiltonian that mixes the X^s and Y^s states. As could be easily checked, the physical states are nonorthogonal; this can be traced back to the non-Hermitian character of the Hamiltonian.

The purpose of this paper is to demonstrate that the calculation of the full S matrix amplitude for a process involving the production and decay of mixed resonances leads to a different mixing prescription for the unstable quasi-physical states than the one obtained from the diagonalization of the effective $M - \mathrm{i}\Gamma/2$ Hamiltonian. In other words, the inclusion of symmetry breaking in the evaluation of transition amplitudes involving the approximation where resonances are described by asymptotic states can be properly done by using the quasi-physical states as given below in (4) and (5), rather than those in (1) and (2). The numerical impact of using both approaches in the evaluation of symmetry breaking, when extracting truncated physical observables as branching fractions for $B \to Vl\nu$ [6,7], can be very important.

To be more specific, let us consider the S matrix amplitudes of the *full* decay processes $B^+ \to (2\pi, 3\pi)l^+\nu_l$, which are dominated by the intermediate ρ and ω resonances (this example illustrates the main characteristics of a two-level unstable mixed system). We show that a convenient prescription for the physical quantum-mechanical eigenstates should be taken as [7]

$$|\rho\rangle = |\rho^I\rangle + \epsilon'|\omega^I\rangle \ , \qquad (4)$$

$$|\omega\rangle = |\omega^I\rangle + \epsilon''|\rho^I\rangle \ , \qquad (5)$$

in order to evaluate the matrix elements of the truncated processes $B^+ \to (\rho^0,\ \omega)l^+\nu_l$ in the presence of isospin symmetry breaking. In the above equations, ϵ' and ϵ'' are given by

$$\epsilon' = \frac{m_{\rho\omega}^2}{m_\rho^2 - m_\omega^2 + \mathrm{i}m_\omega\Gamma_\omega}\ , \qquad (6)$$

$$\epsilon'' = \frac{m_{\rho\omega}^2}{m_\omega^2 - m_\rho^2 + \mathrm{i}m_\rho\Gamma_\rho}\ , \qquad (7)$$

where $m_{\rho\omega}^2 \equiv \langle\omega^I|H^{\Delta I=1}|\rho^I\rangle$ is the $\rho-\omega$ mixing strength. This results in sizable numerical differences, with respect to (1–3), in the evaluation of isospin symmetry-breaking effects, as is discussed in [7].

Let us consider the full S-matrix amplitude for the semileptonic process $B^+(p_B) \to \pi^+(p_1)\pi^-(p_2)l^+(p)\nu_l(p')$, where p_i denotes the corresponding four momenta (the results for the $3\pi l\nu_l$ decay mode are straightforward). Including the contributions of intermediate isospin eigenstates $(\rho^I,\ \omega^I)$ and isospin-breaking effects through $\rho-\omega$ mixing [8], we obtain (we assume that only the ρ^I can couple to the $\pi\pi$ system, i.e., we ignore a possible *direct* contribution $\omega^I \to \pi^+\pi^-$):

$$\begin{aligned}\mathcal{M}(B \to 2\pi l\nu) = \frac{G_F V_{ub}}{\sqrt{2}} l^\mu \{&\mathcal{M}_{\mu\alpha}(B^+ \to \rho^{I*})(\mathcal{P}_\rho)^{\alpha\beta}(q) \\ &+\mathcal{M}_{\mu\alpha}(B^+ \to \omega^{I*})(\mathcal{P}_\omega)^\alpha_\nu(q) \times \mathrm{i}m_{\rho\omega}^2 \\ &\times (\mathcal{P}_\rho)^{\nu\beta}(q)\}\,\mathrm{i}g_{\rho\pi\pi}(p_1-p_2)_\beta\ . \end{aligned} \qquad (8)$$

Here G_F is the Fermi constant, V_{ub} is the relevant CKM matrix element, $g_{\rho\pi\pi}$ is the $\rho\pi\pi$ coupling, l^μ is the leptonic current and $q^2 \equiv (p_1+p_2)^2$ is the squared invariant mass of the 2π system. The hadronic weak matrix element is given by (since we neglect the lepton masses, we drop the terms proportional to $(p+p')_\mu$) [9]:

$$\begin{aligned}\mathcal{M}_{\mu\alpha}(B \to V^*) = &\frac{2}{\Sigma}\epsilon_{\mu\alpha\rho\sigma}p_B^\rho q^\sigma V(t) \\ &+\mathrm{i}\left\{g_{\mu\alpha}\Sigma A_1(t) - \frac{Q_\alpha}{\Sigma}(p_B+q)_\mu A_2(t)\right\}, \end{aligned} \qquad (9)$$

where $\Sigma \equiv m_B + m_V$, $Q = p_B - q$ $(t = Q^2)$ and $V(t),\ A_i(t)$ are Lorentz-invariant form factors. The * symbol means that the vector meson is produced off its mass shell.

The propagators of the resonances are given by:

$$(\mathcal{P}_i)^{\alpha\beta}(q) = \frac{-\mathrm{i}g^{\alpha\beta}}{q^2 - m_i^2 + im_i\Gamma_i} + (\text{terms in } q^\alpha q^\beta). \qquad (10)$$

Since the ρ^I coupling to $\pi^+\pi^-$ is a conserved effective current, i.e., $q\cdot(p_1-p_2) = 0$, only the transverse component of the vector meson propagators gives a non-zero contribution. In addition, because the intermediate ρ^I and ω^I mesons are produced from the recombination of the daughter-$\bar{u}$ (in the $\bar{b} \to \bar{u}$ transition) and the spectator-u quarks, the hadronic weak amplitudes are related by $\mathcal{M}_{\mu\alpha}(B^+ \to \omega^I) = \mathcal{M}_{\mu\alpha}(B^+ \to \rho^I)$. Thus (8) can be written as:

$$\begin{aligned}&\mathcal{M}(B^+ \to 2\pi l\nu) \\ &= \mathrm{i}\frac{G_F V_{ub}}{\sqrt{2}} l^\mu \mathcal{M}_{\mu\alpha}(B^+ \to \rho^{I*}) \times \frac{g^{\alpha\beta}}{q^2 - m_\rho^2 + \mathrm{i}m_\rho\Gamma_\rho} \\ &\quad\times\left\{1 + \frac{m_{\rho\omega}^2}{q^2 - m_\omega^2 + \mathrm{i}m_\omega\Gamma_\omega}\right\} \times \mathrm{i}g_{\rho\pi\pi}(p_1-p_2)_\beta\ . \end{aligned} \qquad (11)$$

A straightforward computation of the 2π invariant mass distribution leads to

$$\begin{aligned}&\frac{\mathrm{d}\Gamma(B^+ \to 2\pi l\nu)}{\mathrm{d}q^2} \\ &= \frac{\sqrt{q^2}}{\pi}\frac{\Gamma(B^+ \to \rho^I(q^2)l^+\nu) \times \Gamma(\rho^I(q^2) \to \pi^+\pi^-)}{(q^2-m_\rho^2)^2 + m_\rho^2\Gamma_\rho^2} \\ &\quad\times\left|1 + \frac{m_{\rho\omega}^2}{q^2 - m_\omega^2 + \mathrm{i}m_\omega\Gamma_\omega}\right|^2 . \end{aligned} \qquad (12)$$

The q^2 in the argument of ρ^I means that decay widths must be taken with the ρ^I off its mass shell.

A very similar evaluation of the 3π mass distribution in the decay $B^+ \to \pi^+\pi^-\pi^0 l^+\nu_l$ gives (in this case, $q^2 = (p_1+p_2+p_3)^2$ corresponds to the 3π invariant mass):

$$\begin{aligned}&\frac{\mathrm{d}\Gamma(B^+ \to 3\pi l\nu)}{\mathrm{d}q^2} \\ &= \frac{\sqrt{q^2}}{\pi}\frac{\Gamma(B^+ \to \omega^I(q^2)l^+\nu) \times \Gamma(\omega^I(q^2) \to \pi^+\pi^-\pi^0)}{(q^2-m_\omega^2)^2 + m_\omega^2\Gamma_\omega^2} \\ &\quad\times\left|1 + \frac{m_{\rho\omega}^2}{q^2 - m_\rho^2 + \mathrm{i}m_\rho\Gamma_\rho}\right|^2 . \end{aligned} \qquad (13)$$

The factorization of the decay widths in (12) and (13) is an exact result that follows from the conserved effective current conditions in the $\rho \to 2\pi$ and $\omega \to 3\pi$ vertices.

The quasi-physical on-shell decay widths of the $B^+ \to \rho l^+\nu$ and $B^+ \to \omega l^+\nu$ decays are obtained by fixing the 2π- and the 3π-invariant masses, at the ρ and ω meson masses, respectively (in practice, the cuts $m_V^2 - \Delta < q^2 < m_V^2 + \Delta$ are necessary to isolate the vector mesons from the q^2 distribution). Under these conditions, we get:

$$\begin{aligned}&\left.\frac{\mathrm{d}\Gamma(B^+ \to 2\pi l\nu)}{\mathrm{d}q^2}\right|_{q^2=m_\rho^2} \\ &= \frac{1}{\pi m_\rho\Gamma_\rho}\Gamma(B^+ \to \rho^I l\nu) \times B(\rho^I \to 2\pi)|1+\epsilon'|^2, \end{aligned} \qquad (14)$$

$$\begin{aligned}&\left.\frac{\mathrm{d}\Gamma(B^+ \to 3\pi l\nu)}{\mathrm{d}q^2}\right|_{q^2=m_\omega^2} \\ &= \frac{1}{\pi m_\omega\Gamma_\omega}\Gamma(B^+ \to \omega^I l\nu) \times B(\omega^I \to 3\pi)|1+\epsilon''|^2\ . \end{aligned} \qquad (15)$$

Therefore, as already pointed out in [7], the isospin-breaking effects through ϵ', ϵ'' must be removed from the measured invariant-mass distributions quoted in [6], in order to compare quantities related by isospin symmetry. The results given in (14) and (15) are identical to the ones obtained in [7], where it is assumed that the physical quantum mechanical eigenstates for the ρ^0 and ω mesons are given by (4) and (5).

Another way to compare the symmetry-breaking effects in semileptoic B decays arising from the prescriptions of (1–3) and (4–5) is to decompose the resonant pieces of the amplitudes for 2π and 3π semileptonic B decays, respectively. This gives:

$$\frac{1}{s_\rho}\left\{1+\frac{m^2_{\rho\omega}}{s_\omega}\right\} = \frac{1}{s_\rho}\left\{1+\frac{m^2_{\rho\omega}}{\delta^2}\right\} - \frac{1}{s_\omega}\times\frac{m^2_{\rho\omega}}{\delta^2}, \quad (16)$$

$$\frac{1}{s_\omega}\left\{1+\frac{m^2_{\rho\omega}}{s_\rho}\right\} = \frac{1}{s_\omega}\left\{1-\frac{m^2_{\rho\omega}}{\delta^2}\right\} + \frac{1}{s_\rho}\times\frac{m^2_{\rho\omega}}{\delta^2}, \quad (17)$$

where $s_V \equiv q^2 - m_V^2 + \mathrm{i} m_V \Gamma_V$ and $\delta^2 \equiv m_\rho^2 - m_\omega^2 + \mathrm{i}(m_\omega \Gamma_\omega - m_\rho \Gamma_\rho) \approx 2\bar{m}\{m_\rho - m_\omega + \mathrm{i}(\Gamma_\omega - \Gamma_\rho)/2\}$, and $\bar{m}$ is the average mass of ρ and ω mesons. Note that the first term in the right-hand side (r.h.s.) of (16) and (17) corresponds to (1–3) and gives equal strengths for isospin breaking in the $B^+ \to (\rho^0,\ \omega) l^+ \nu$ decay rates. However, the second terms in the r.h.s. of (16) and (17) give very different contributions, due to the propagation of the ω (ρ) meson in the 2π (3π) channel.

From (14) and (15), we see that the effects of isospin breaking in $B^+ \to \rho^0 l^+ \nu$ are more important than in the $B^+ \to \omega l^+ \nu$ transition, because $|1+\epsilon'| \approx 1.18$, $|1+\epsilon''| \approx 1.0$. This fact is accidental, because $m_\omega - m_\rho \approx \Gamma_\omega$, and therefore the real and imaginary parts in ϵ', have almost equal weights. The situation is quite similar in the $K_L - K_S$ system where $m_{K_L} - m_{K_S} \approx (\Gamma_{K_S} - \Gamma_{K_L})/2 \approx \Gamma_{K_S}/2$; therefore, there is not an important numerical difference when computing mixing effects in $K_L \to 2\pi$ decays through (1)–(3) or (4) and (5). However, the effects are different in CP-violating $K_S \to 3\pi$ decays. As is well known (see [4,10]), the mixing of states accounts for the complex phase ($\approx \pi/4$) in the CP violation parameters $\eta_{+-,00}$ measured in $K_L \to \pi\pi$ decays. Using the same prescription as the one for the $\rho - \omega$ system, we would find that (4) and (5) imply that the complex phase in CP-violating parameters of $K_S \to 3\pi$ decays should be almost zero, which is in clear disagreement with the results obtained using the conventional quantum-mechanical eigenstates of (1)–(3), which predict the same phase as in $K_L \to 2\pi$.

In practice, however, it is difficult to test the difference between both approaches as far as $B^+ \to \omega l \nu$ and $K_S \to 3\pi$ are concerned. On the one hand, CP violation (and therefore the complex phase of $\eta_{+-0,000}$) has not been observed yet in $K_S \to 3\pi$ decays [the testing of whether the mixing of unstable states is given accurately by (1)–(3) or (4) and (5) is not possible]. A similar unfortunate situation is present in the $\rho - \omega$ system, because the very narrow width of the ω meson does not allow one to show, by a fine scanning of the $e^+e^- \to \pi^+\pi^-\pi^0$ cross section in the $\rho - \omega$ region, as done in $e^+e^- \to \pi^+\pi^-$ [11], the interference effects due to $\rho - \omega$ mixing.

In conclusion, a consistent treatment of unstable particles, as provided by quantum field theory, leads to a different mixing scheme for quasi-physical states of a two-level unstable system than the one obtained from the traditional approach based on an $M - \mathrm{i}\Gamma/2$ non-Hermitian effective Hamiltonian. Symmetry-breaking effects in truncated observables, such as isospin violation in semileptonic $B^+ \to (\rho^0,\ \omega)$ transitions, or CP violation in $K_L - K_S$ decays, turn out to be very different in both approaches.

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